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On the KMS condition for Z_{2N} chiral conformal field theories

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Abstract. Using the Kubo–Martin–Schwinger (KMS) condition and exchange algebras we discuss how to compute correlation functions for the conformal exchange field theories. In particular, we derive the Verlinde algebra from modular invariance of the Z_{2N} conformal field theories. We also show how the Verlinde matrices can be related to the KMS matrices.

1. Introduction

The concept of exchange algebra has been introduced in the context of two-dimensional conformal field theories to characterize the algebraic structure of the light-cone interpolating fields. More precisely it states how these fields can be braided. The braid matrices satisfy simple equations, one of which being a Yang–Baxter type equation. Rehren [1] showed with a simple calculation how these structural equations contain a good deal of information. Indeed, using an additional hypothesis, he was able to calculate the braiding properties of some basic fields and to obtain the spectrum of both the minimal models and WZW theories. His hypothesis was the existence of a field α whose fusion rules, for some labelling of the conformal families, are $[\alpha][l] = [l-1] \oplus [l+1]$ for $l \in Z$.

In the present paper we apply these ideas to another set of simple CFTs namely the Z_{2N} conformal field theories [2].

The outline of this paper is as follows. In section 2 we present the Rehren and Schroer's definition of exchange algebra [3] and Buchholz–Mack–Todorov definition of the KMS condition in the context of conformal quantum field theories [4]. In section 3 we apply these ideas to the Z_{2N} chiral conformal field theories. Finally in section 4 we explain the relationship between the KMS matrices and those satisfying the Verlinde algebra.

2. Exchange algebra and the KMS condition in conformal quantum field theories

2.1. Exchange algebra in conformal quantum field theories

Due to the non-additivity of conformal scale dimensions, the spectrum decomposition of local fields $\Phi(\boldsymbol{x})$, with respect to the centre of the conformal group is non-trivial [5],

$$\Phi(\boldsymbol{x}) = \sum_{\eta} \Phi_{\eta}(\boldsymbol{x}) \tag{2.1}$$

where every $\Phi_{\eta}(\boldsymbol{x})$ is a non-local object with η-dependent complex phases that occur in the special conformal transformation laws. The range of the label η is determined by the selection rules of scale dimensions.

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Applied to the vacuum state, the fields of a conformal block $[\alpha]$ generate a representation sector H_α of the stress–energy tensor field. Applied to a state in H_β, fields of a conformal block $[\alpha]$ give us contributions in all space H_γ allowed by the fusion rules. Introducing orthogonal projectors P_β on the sectors H_β one obtains the decomposition

$$\Phi^\alpha(\boldsymbol{x}) = \sum_{\beta,\gamma} P_\gamma \Phi^\alpha(\boldsymbol{x}) P_\beta \equiv (\Phi^\alpha)_{\gamma\beta}(\boldsymbol{x}). \tag{2.2}$$

This decomposition coincides with the spectral decomposition (2.1) with the previous label η replaced 'fusion channels' for the 'charge' α.

It is a well established fact that conformal field theories can be constructed on Hilbert spaces which are direct sums of irreducible representations of an observable algebra $A \oplus \bar{A}$. Both subalgebras $A \oplus \mathbf{1}$ and $\mathbf{1} \oplus \bar{A}$ are associated to one light-cone and are unitary. We also add the further requirement that the Hilbert space contains only a finite number of irreducible representations of A and $\bar{A}$. Hence

$$H = \bigoplus_{\alpha,\bar{\alpha}} H_\alpha \otimes H_{\bar{\alpha}} \tag{2.3}$$

where H_α ($H_{\bar{\alpha}}$) are irreducible representations of A ($\bar{A}$) and the pair $(\alpha, \bar{\alpha})$ takes its values in a finite set.

Due to the light-cone factorization of the stress–energy tensor field algebra, the label $[\alpha]$ of conformal blocks are in fact pairs $[\alpha_+, \alpha_-]$. Both representation sectors and the projectors factorize into the projected fields

$$(\Phi^\alpha)_{\gamma\beta}(\boldsymbol{x}) = (A^{\alpha_+})_{\gamma_+\beta_+}(\boldsymbol{x}_+) \otimes (A^{\alpha_-})_{\gamma_-\beta_-}(\boldsymbol{x}_-). \tag{2.4}$$

Finally, the monodromy properties of the conformal blocks are equivalent to the exchange algebra on either light-cone

$$(A^{\alpha_1})_{\delta\gamma}(x)(A^{\alpha_2})_{\gamma\beta}(y) = \sum_{\gamma'} [R^{(\delta,\beta)}_{(\alpha_1,\alpha_2)}(s)]_{\gamma\gamma'} (A^{\alpha_2})_{\delta\gamma'}(y)(A^{\alpha_1})_{\gamma'\beta}(x). \tag{2.5}$$

Here (and from now on) we have omitted the indices '$\pm$'. The numerical structure constants R are matrices which satisfy three basic properties (see [3]).

(i) $[R^{(\delta,\beta)}_{(\alpha_1,\alpha_2)}(s)]$ depend on x and y only through their relative position. This follows from translation and scale variance. Moreover, if $s = \mathrm{sign}(x - y) = \pm$, then

$$[R^{(\delta,\beta)}_{(\alpha_1,\alpha_2)}(+)]^{-1} = R^{(\delta,\beta)}_{(\alpha_1,\alpha_2)}(-). \tag{2.6}$$

(ii) *Phase condition*. $R^{(\delta,\beta)}_{(\alpha_1,\alpha_2)}$ and $R^{(\delta,\beta)}_{(\alpha_2,\alpha_1)}$ are related through the following relation

$$\sum_{\gamma'} [R^{(\delta,\beta)}_{(\alpha_1,\alpha_2)}(s)]_{\gamma\gamma'} [R^{(\delta,\beta)}_{(\alpha_2,\alpha_1)}(s)]_{\gamma'\gamma''} \exp(2\mathrm{i}\pi(h_\gamma + h_{\gamma'} - h_\delta - h_\beta) = \delta_{\gamma,\gamma''} \tag{2.7}$$

where h_γs are primary dimensions of the representations $[\gamma]$. This follows from invariance under special conformal transformation.

(iii) *Braid relations*. The exchange matrices satisfy

$$\begin{aligned} &\sum_{\beta_1''} [R^{(\beta_0,\beta_2)}_{(\alpha_1,\alpha_2)}(s)]_{\beta_1\beta_1''} [R^{(\beta_1'',\beta_3)}_{(\alpha_1,\alpha_3)}(s)]_{\beta_2\beta_2''} [R^{(\beta_0,\beta_2')}_{(\alpha_2,\alpha_3)}(s)]_{\beta_1''\beta_1'} \\ &\qquad = \sum_{\beta_2''} [R^{(\beta_1,\beta_2)}_{(\alpha_2,\alpha_3)}(s)]_{\beta_2\beta_2''} [R^{(\beta_0,\beta_2'')}_{(\alpha_1,\alpha_3)}(s)]_{\beta_1\beta_1'} [R^{(\beta_1',\beta_3)}_{(\alpha_1,\alpha_2)}(s)]_{\beta_2''\beta_2'} \end{aligned} \tag{2.8}$$

which is the consistency relation for the associativity of the exchange algebra (2.5).

All these relations were derived in [6] from the theory of localized endomorphism without invoking conformal invariance.

2.2. The KMS condition in conformal quantum field theories

Here we shall exploit the well known correspondence between finite size Euclidean theories and finite positive temperature quantum field theories in a Minkowski space to formulate a physical condition, namely the Kubo–Martin–Schwinger (KMS) boundary condition. The KMS condition can be used to evaluate finite temperature correlation functions of fields which satisfy simple algebraic commutation relations. In the case of conformal fields it can be combinated with a small distance operator product to compute the partition function of the system.

In the context of conformal field theories, the KMS condition was first explored by Buchholz *et al* [4].

A Gibbs equilibrium state of inverse temperature β, $\langle\circ\rangle_\beta$ (KMS state), is characterized in the following way. For any two elements A and B of a field algebra and for a given time evolution automorphism α_t, the functions $\langle A\alpha_t(B)\rangle_\beta$ and $\langle\alpha_t(B)A\rangle_\beta$ with $t \in R$ can be regarded as boundary values of an analytic function $\langle A\alpha_\zeta(B)\rangle_\beta$. This function is holomorphic in the strip $0 < \mathrm{Im}\,\zeta < \beta$, due to energy positivity, and satisfies the KMS boundary condition

$$\langle A\alpha_{t+\mathrm{i}\beta}(B)\rangle_\beta = \langle\alpha_t(B)A\rangle_\beta \tag{2.9}$$

where the time-evolution automorphism α_t is defined by

$$\alpha_t(A) = \mathrm{e}^{\mathrm{i}Ht}A\mathrm{e}^{-\mathrm{i}Ht}. \tag{2.10}$$

If $\mathrm{e}^{-\beta H}$ is a trace-class operator, in the sense that the partition function

$$Z = \mathrm{Tr}(\mathrm{e}^{-\beta H}) \tag{2.11}$$

is well defined, then $\langle A\rangle_\beta$ is given by the density matrix

$$\langle A\rangle_\beta = \frac{1}{Z(\beta)}\,\mathrm{Tr}(\mathrm{e}^{-\beta H}A). \tag{2.12}$$

H is bounded below and the trace is carried over the eigenstates of H.

For conformal field models, H will be replaced by the conformal Hamiltonian $L_0 + \bar{L}_0$ and we will be dealing with representations for which the trace (2.11) does exist.

Relation (2.9) can be used for the computation of the possible finite-temperature states if there is a simple algebraic relation between the operators $\alpha_t(B)A$ and $A\alpha_t(B)$. For example, let the commutator of these operators be a multiple of the identity $C(t)\mathbf{1}$. Then one rewrites relation (2.9) according to

$$\langle A\alpha_{t+\mathrm{i}\beta}(B)\rangle_\beta = \langle[\alpha_t(B), A]\rangle_\beta + \langle A\alpha_t(B)\rangle_\beta = C(t) + \langle A\alpha_t(B)\rangle_\beta \tag{2.13}$$

where we have used the fact that $\langle\mathbf{1}\rangle_\beta = 1$. Consequently one obtains an inhomogeneous functional equation for the function $t \to \langle A\alpha_t(B)\rangle_\beta$, and this equation can be solved straightforwardly by Fourier transformation [7].

In the case of the Virasoro algebra or Kac–Moody algebras the commutator of the basic fields is not a c-number, but it is linear in these fields. Therefore, the first equation in (2.13) provides a recursive relation between the n-point function and the $(n-1)$-point functions which can likewise be solved for these models.

If A and B are commuting operators (bosonic at different localization points) the KMS condition becomes simply the periodic condition in the imaginary direction. For operators with braid-group commutation relations, the KMS condition leads to a non-trivial matrix ‘quasi-periodic’.

In the light-cone field theory we have a simple algebraic relation between the operators $A\alpha_t(B)$ and $\alpha_t(B)A$, the exchange algebra for chiral fields [1], so one obtains the KMS condition on a light-cone. To be more explicit let us use the compact picture [8].

The compact picture of the stress–energy tensor, say $T(x_+) \to \mathcal{T}(\alpha)$ $(x_+ = 2\mathrm{i}\tan(\pi\alpha))$, has the following Laurent expansion:

$$\mathcal{T}(\alpha) = \sum_{n\in Z} \tilde{L}_n \mathrm{e}^{-2\mathrm{i}\pi n\alpha} \qquad \tilde{L}_n = L_n - \frac{c}{24}\delta_{n,0}. \tag{2.14}$$

In order to compute the KMS states one can use the factorized form of the local field $\Phi(\alpha, \bar{\alpha})$ in terms of interpolating fields, equation (2.4). This is done by considering the 2-point function $\langle \Phi(\alpha, \bar{\alpha})\Phi(0, 0)\rangle_\beta$ as a product of two chiral $\langle A(\alpha)A(0)\rangle_\beta$ and $\langle A(\bar{\alpha})A(0)\rangle_\beta$ KMS states, i.e.

$$\langle A(\alpha)A(0)\rangle_\beta = \frac{\mathrm{Tr}(\mathrm{e}^{-\beta\tilde{L}_0}A(\alpha)A(0))}{\tilde{Z}(\beta)} \tag{2.15}$$

where $\tilde{Z}(\beta)$ is the chiral partition function

$$\tilde{Z}(\beta) = \mathrm{Tr}(\mathrm{e}^{-\beta\tilde{L}_0}) \qquad \tilde{L}_0 = L_0 - \frac{c}{24}. \tag{2.16}$$

Using the orthogonal projectors we can write

$$\begin{aligned}\tilde{Z}(\beta)\sum_{\lambda,\lambda'}\langle P_\lambda A(\alpha)P_{\lambda'}A(0)P_\gamma\rangle &= \sum_{\lambda,\lambda'}\mathrm{Tr}(\mathrm{e}^{2\mathrm{i}\pi\tau\tilde{L}_0}P_\lambda A(\alpha)P_{\lambda'}A(0)) \\ &= \sum_{\lambda,\lambda'}\mathcal{F}_{\lambda\lambda'}(\alpha \mid \tau)\end{aligned} \tag{2.17}$$

where we have introduced a new variable τ through the relation $2\mathrm{i}\pi\tau = -\beta$.

In the compact picture the analytically continued time evolution automorphism of the field $A(\alpha)$ is given by

$$\alpha_{i\beta}(A(\alpha)) = A(\alpha + \tau). \tag{2.18}$$

Therefore we can write the KMS condition as

$$\sum_{\lambda,\lambda'}\mathrm{Tr}(\mathrm{e}^{2\mathrm{i}\pi\tau\tilde{L}_0}P_\lambda A(\alpha+\tau)P_{\lambda'}A(0)) = \sum_{\lambda,\lambda'}\mathrm{Tr}(\mathrm{e}^{2\mathrm{i}\pi\tau\tilde{L}_0}P_{\lambda'}A(0)P_\lambda A(\alpha)). \tag{2.19}$$

In order to exhibit the functional equations for the chiral KMS states we use the exchange algebra (2.5) and equation (2.17) to write (2.19) as

$$\mathcal{F}_{\lambda\lambda'}(\alpha+\tau|\tau) = \sum_{\lambda''}[R^{(\lambda',\lambda')}(\pm)]_{\lambda\lambda''}\mathcal{F}_{\lambda\lambda''}(\alpha|\tau) \tag{2.20}$$

where the indices λ, λ', λ'' are labelling irreducible representations (sectors) of some chiral algebra which satisfy certain fusion rules.

Next, we consider the action of the centre of the conformal group, generated by $Z = \exp(2\mathrm{i}\pi L_0)$, on the primary field $A(\alpha)$ [3]

$$ZP_\lambda A(\alpha)P_{\lambda'}Z^{-1} = \mathrm{e}^{2\mathrm{i}\pi(h_{\lambda'}-h_\lambda)}P_\lambda A(\alpha)P_{\lambda'}. \tag{2.21}$$

This provides us with another set of functional equations for the KMS states

$$\mathcal{F}_{\lambda\lambda'}(\alpha+1|\tau) = \mathrm{e}^{2\mathrm{i}\pi(h_{\lambda'}-h_\lambda)}\mathcal{F}_{\lambda\lambda'}(\alpha|\tau). \tag{2.22}$$

Therefore, knowing the exchange matrices R and the conformal dimensions h_λ, we can (in principle) compute the KMS states by solving the functional equations (2.20) and (2.22).

3. The Z_{2N} chiral conformal model—a simple application

This model is perhaps the simplest chiral conformal quantum field theory with non-trivial modular properties. Here we will consider it as a simple illustration of the applicability of the program described in the previous section.

3.1. Z_{2N} exchange algebra

There exists a natural way to generate some fusion rules from finite group theory [9]. Let us consider G a finite group, it has a finite number of inequivalent irreducible representations which we denote by Π_α. Any tensor product of them can be decomposed into irreducible representations

$$\Pi_\alpha \otimes \Pi_\beta = \oplus_\gamma (N_\alpha)^\gamma_\beta \Pi_\gamma \tag{3.1}$$

where $(N_\alpha)^\gamma_\beta$ is the multiplicity of the Π_γ representation. Thanks to the associativity and commutativity of the tensor product, the N_α's define a representation of a commutative and associative algebra, the fusion algebra. In the corresponding field theory, with each irreducible representation of G, we associate a primary field such that $(N_\alpha)^\gamma_\beta$ of finite group theory gives us the fusion algebra of the underlying field theory. Naturally, the trivial representation of G is associated with the identity operator and the conjugate irreducible representations of G are associated with conjugate fields. In this case we obtain a special family of braid group representations with integer statistical dimensions, but possible non-trivial statistic phases. In conformal models, the statistical dimensions are known as the normalized entries S_{0p}/S_{00} of the modular matrix, measuring the relative dimension of the representation of the chiral algebra [10, 11].

If G is a finite Abelian group (say, Z_N group) its irreducible representations are one-dimensional and are labelled by the elements of Z_N. In this case, $(N_\alpha)^\gamma_\beta = \delta^\gamma_{\alpha+\beta}$ with $\alpha, \beta, \gamma = 0, 1, \ldots, N-1$. Thus the fusion algebra is

$$[\alpha][\beta] = [\alpha + \beta]. \tag{3.2}$$

Now consider the following set of conformal families $F = [\Phi_\alpha]$, $\alpha = 0, 1, \ldots, N-1$, together with the following fusion rules $[\Phi_{v_1}][\Phi_{v_2}] = [\Phi_{v_1+v_2}]$. Any family of F is generated by the operator product expansion of fields belonging to the family $[\Phi_1]$. The exchange algebra of any field in F can then be obtained from those of the field Φ_1, which plays a similar role to the field $\Phi_{(1,2)}$ in the $SU(2)$ minimal models considered by Rehren [1].

Let us now introduce the notation $A_{v+\alpha,\alpha}(x) = P_{v+\alpha}A(x)P_v$ for the intertwining field between the sectors $[v]$ and $[v+\alpha]$, where $v = 0, 1, \ldots, N-1$ and $\alpha \in Z$. Now we recall (2.5) and define the following exchange algebra

$$\begin{aligned} P_{v_0}A(x)P_{v_0+\alpha_1}A(y)P_{v_0+\alpha_1+\alpha_2} \\ = \sum_{v'}[R^{(v_0,v_0+\alpha_1+\alpha_2)}_{(\alpha_1,\alpha_2)}(\pm)]_{v_0+\alpha_1,v'}P_{v_0}A(y)P_{v'}A(x)P_{v_0+\alpha_1+\alpha_2}. \end{aligned} \tag{3.3}$$

The fusion rules (3.2) imply $\alpha_1 = -\alpha_2 = \alpha$ and the sum in v' has only one term, i.e. $v' = v_0 + \alpha_2$ and we have for the elementary field ($\alpha = 1$) the following R matrices

$$[R^{(v,v)}_{(1,\bar{1})}]_{v-1,v+1} = [R^{(v,v)}_{(\bar{1},1)}]_{v+1,v-1} = \eta \tag{3.4}$$

for $v = 1, 2, \ldots, N-2$. For $v = 0$ and for $v = N-1$ we define

$$[R^{(0,0)}_{(1,\bar{1})}]_{1,1} = [R^{(N-1,N-1)}_{(1,\bar{1})}]_{N-2,N-2} \doteq \eta \tag{3.5}$$

where η is a phase factor which will be found from the phase condition (2.7) and the normalization of the 2-point function. The braid matrices do not depend on v and the phase condition can be written as

$$\exp(2\mathrm{i}\pi(h_{v-1} + h_{v+1} - 2h_v)) = \eta^{-2} \tag{3.6}$$

which yields the constraint

$$\eta^{-2N} = 1. \tag{3.7}$$

From the 2-point function $A_{01}(x)A_{10}(y) \doteq \eta A_{01}(y)A_{10}(x)$, we have

$$\eta^{-1} = \exp(2\mathrm{i}\pi h_1) \tag{3.8}$$

and we also impose that $h_0 = 0$ and $h_{N-v} = h_v$. So the general solution of (3.6) with the constraint (3.7) is given by

$$h_v = v^2 h_1 = \frac{v^2}{4N} \qquad v = 0, 1, \ldots, 2N - 1. \tag{3.9}$$

A systematic investigation of the full Z_N spectrum was made by Degiovanni [2]. He showed that the solutions (3.9) can also be derived from the equation $(ST)^3 = 1$, where S and T are matrices representing the modular group, in agreement with Verlinde algebra [12].

Now we can discuss a solution to the KMS condition equations (2.20) and (2.22), which turns out to be the KMS states for Z_{2N} chiral conformal field theories. From the exchange algebra for Z_{2N} models equations (3.3) and (3.4), we have the following equations,

$$\begin{aligned} &\mathcal{F}_{\lambda,\lambda+1}(\alpha + \tau|\tau) = \eta\mathcal{F}_{\lambda+1,\lambda+2}(\alpha|\tau) \\ &\mathcal{F}_{\lambda,\lambda+1}(\alpha + 1|\tau) = \exp(2\mathrm{i}\pi(h_{\lambda+1} - h_\lambda))\mathcal{F}_{\lambda,\lambda+1}(\alpha|\tau) \end{aligned} \tag{3.10}$$

where $\lambda = 0, 1, \ldots, N - 1$ and $\mathcal{F}_{N-1,N}(\alpha|\tau) = \mathcal{F}_{N-1,0}(\alpha|\tau)$, and the dimensions h_v are given by (3.9). We now proceed to the solution of these functional equations.

$$\begin{aligned} &\mathcal{F}_{\lambda,\lambda+1}(\alpha|\tau) = \frac{1}{\eta(\tau)}\left[2\mathrm{i}\pi\frac{\theta_1(\alpha|\tau)}{\theta_1'(0|\tau)}\right]^{-1/2N}\sum_{l\in Z} q^{N(l+\lambda/2N)^2}\mathrm{e}^{2\mathrm{i}\pi(l+\lambda/2N)\alpha} \\ &\mathcal{F}_{N,N+1}(\alpha|\tau) = \mathcal{F}_{N,1-N}(\alpha|\tau) \end{aligned} \tag{3.11}$$

where $\theta_1(\alpha|\tau)$ $(\theta_1'(\alpha|\tau) = \partial\theta_1(\alpha|\tau)/\partial\alpha)$ is the first Jacobi theta function, $\eta(\tau)$ is the Dedekind η-function and $q = \mathrm{e}^{2\mathrm{i}\pi\tau}$. Here we have used the fact that h_v (3.9) is defined mod $2N$ in order to label the Z_{2N} irreducible representations by $\lambda = 1 - N, 2 - N, \ldots, N$. Equations (3.10) can also be written in a compact matricial form

$$\mathcal{F}(\alpha + \tau|\tau) = \tilde{S}\mathcal{F}(\alpha|\tau) \qquad \mathcal{F}(\alpha + 1|\tau) = \tilde{T}\mathcal{F}(\alpha|\tau) \tag{3.12}$$

where the chiral components $\mathcal{F}_{\lambda,\lambda+1}$ are indexed as elements of the matrix column $\mathcal{F}$ and

$$\begin{aligned} &\tilde{S} = (\tilde{S})_{\lambda\lambda'} = \eta\delta_{\lambda,\lambda'+1} \qquad \tilde{T} = (\tilde{T})_{\lambda\lambda'} = \eta^{-1}\omega^\lambda\delta_{\lambda,\lambda'} \\ &\lambda, \lambda' = 1 - N, 2 - N, \ldots, N. \end{aligned} \tag{3.13}$$

Both $\tilde{S}$ and $\tilde{T}$ matrices are defined mod $2N$. Note also that $\eta^{-1}\tilde{S}$, $\eta\tilde{T}$ and $\omega = \exp(-\mathrm{i}\pi/N)$ define a representation of the Heisenberg group similar to that used by Capelli–Itzykson–Zuber [11] and Gepner–Qiu [13]. We shall call $\tilde{S}$ and $\tilde{T}$ as KMS matrices for the Z_{2N} chiral conformal models.

In [4], Buchholz, Mack and Todorov (BMT) derived a way of computing the characters of the Z_{2N} models by exploiting the KMS condition for Gibbs states in the algebra of fields or observables.

Here we can only sketch the comparisons of (3.11) with the KMS 2-point function $\omega_\beta(\Psi_{-g}(0)\Psi_g(t))$ on the algebra $\mathcal{A}_N$ where $\Psi_g \in \mathcal{A}_N$ is any unitary field operator with charge $g = \rho[1] = \sqrt{2N}$. The reader can find the details in [4].

Using the BMT approach we derived the following KMS 2-point function

$$\omega_\beta(\Psi_{-g}(0)\Psi_g(t))_\lambda = \left[2\mathrm{i}\pi \frac{\theta_1(\alpha|\tau)}{\theta_1'(0|\tau)}\right]^{-g^2} \frac{\Theta_{2\lambda,g^2}(tg,\tau,0)}{\Theta_{2\lambda,g^2}(0,\tau,0)} \tag{3.14}$$

where we used the classical Θ-function

$$\Theta_{l,g^2}(t,\tau,u) = \mathrm{e}^{2\mathrm{i}\pi u}\sum_{n\in Z} \mathrm{e}^{\mathrm{i}\pi\tau(ng+l/g)^2+2\mathrm{i}\pi t(ng+l/g)}.$$

We can proceed by taking the limit $t \to 0$ of (3.14), which gives

$$\omega_\beta(\Psi_{-g}(0)\Psi_g(t))_\lambda \overset{t\to 0}{\sim} [2\mathrm{i}\sin(t/2)]^{-g^2}\left\{1 + \mathrm{i}gt\frac{\sum_n (ng+\lambda/g)\mathrm{e}^{-\beta(ng+\lambda/g)^2/2}}{\sum_n \mathrm{e}^{-\beta(ng+\lambda/g)^2/2}}\right.$$
$$\left. + g^2t^2\frac{\partial}{\partial\beta}\ln\left[\frac{\sum_n \mathrm{e}^{-\beta(ng+\lambda/g)^2/2}}{\Pi_n(1-\mathrm{e}^{-n\beta})}\right] + \cdots\right\}. \tag{3.15}$$

Comparing (3.15) with the short distance operator product expansion [4],

$$\Psi_{-g}(0)\Psi_g(t) = [2\mathrm{i}\sin(t/2)]^{-g^2}\{1 + [2\mathrm{i}\sin(t/2)]J(t) + [2\mathrm{i}\sin(t/2)]^2\mathcal{T}(t)\}$$

where $\mathcal{T}(t) = L_0 - c/24 + \sum_{n\neq 0} L_n\mathrm{e}^{-nt}$ is the stress tensor, we get

$$\omega_\beta(\mathcal{T}(t)) = \omega_\beta(L_0) = -\frac{\partial}{\partial\beta}\ln\left[\frac{\sum_{n\in Z}\mathrm{e}^{-\beta(ng+\lambda/g)^2/2}}{\Pi_{n=1}^\infty(1-\mathrm{e}^{-n\beta})}\right].$$

On the other hand,

$$\omega_\beta(L_0) = -\frac{\partial}{\partial\beta}\ln[\mathrm{e}^{-\beta/24}Z_\lambda(\beta)]$$

and we obtain the following expression for the chiral partition function:

$$Z_\lambda(\beta) = \frac{\sum_{n\in Z}\mathrm{e}^{-\beta(ng+\lambda/g)^2/2}}{\mathrm{e}^{-\beta/24}\Pi_{n=1}^\infty(1-\mathrm{e}^{-n\beta})}.$$

This result for the partition function is proportional to a ratio of a classical Θ-function and a Dedekind η-function.

Finally we observe that the fields $\Psi_g(t)$ in (3.14) are related to the elementary fields $A(\alpha)$ $(t = 2\mathrm{i}\pi\alpha)$ in (3.11) by a Klein transformation [4], so one can put the complete expression (3.14) into a form which can be compared with the equation (3.11).

4. Verlinde's algebra from KMS modular properties

Before we start to study the modular properties of the KMS states let us recall some basic facts about Verlinde's works [12, 14]. The main tool is the fusion algebra. It is defined as the associative algebra

$$\Phi_i \times \Phi_j = N_{ij}^k\Phi_k \tag{4.1}$$

where the non-negative integer N_{ij}^k is the number of independent couplings of the type (ijk). Since (4.1) is also commutative, it has N irreducible one-dimensional representations $\lambda_i^{(n)}, n = 1,\ldots,N$,

$$\lambda_i^{(n)}\lambda_j^{(n)} = N_{ij}^k \lambda_k^{(n)} \tag{4.2}$$

which are given by the eigenvalues of the matrices $(N_i)_j^k$.

There is a remarkable relation between the fusion algebra and the modular properties of the one-loop characters: the matrix S describing the behaviour under $\tau \to -1/\tau$ diagonalizes the fusion algebra. More precisely it expresses the one-dimensional representation $\lambda_i^{(j)}$ and the integers N_{ijk} in terms of S as

$$\lambda_i^{(j)} = \frac{S_{ij}}{S_{0j}} \qquad \text{and} \qquad N_{ijk} = \sum_n \frac{S_{in}S_{jn}S_{nk}}{S_{0n}}. \tag{4.3}$$

One can refer to [14] for a proof of these relations.

An important application of relation (4.3) concerns the evaluation of possible values of c (central charge) and h (conformal dimension) for a given fusion algebra. We also stress that the condition 'S is symmetric' is absolutely necessary if we want this matrix to represent both the modular properties of some characters and the monodromy matrix [5].

To derive the modular properties of the functions $\mathcal{F}_{\lambda,\lambda+1}(\alpha|\tau)$ for Z_{2N} conformal models we apply the same procedure used in [4].

The $2N$ functions $\mathcal{F}_{\lambda,\lambda+1}(\alpha|\tau)$, $\lambda = 1-N, \ldots, N$ span a $2N$-dimensional representation of the modular group $SL(2, Z)$, which acts on the variables (α, τ) according to

$$(\alpha, \tau) \to (\alpha_G, \tau_G)$$

$$\alpha_G = \alpha \qquad \tau_G = \frac{a\tau + b}{c\tau + d}$$

where G^{-1} is the 2×2 matrix with integer entries $a, b, c, d \in Z$, $ad - bc = 1$.

It is sufficient to consider the transformation properties of $\mathcal{F}_{\lambda,\lambda+1}(\alpha|\tau)$ for the two generators of $SL(2, Z)$

$$T : \tau \to \tau + 1 \qquad S : \tau \to -\frac{1}{\tau} \tag{4.4}$$

satisfying

$$S^2 = (ST)^3. \tag{4.5}$$

First we note that the function appearing in $\mathcal{F}_{\lambda,\lambda+1}(\alpha|\tau)$ is the Dedekind η-function

$$\eta(\tau) = \mathrm{e}^{2\mathrm{i}\pi\tau/24} \prod_{n=1}^{\infty} (1 - \mathrm{e}^{2\mathrm{i}\pi n\tau}) \tag{4.6}$$

and

$$\theta_1(\alpha|\tau) = \sum_{n\in Z} \mathrm{e}^{\mathrm{i}\pi(n-1/2)^2\tau + 2\mathrm{i}\pi(n-1/2)(\alpha-1/2)} \tag{4.7}$$

is the first Jacobi θ-function. Consequently, under the transformation T

$$\mathcal{F}_{\lambda,\lambda+1}(\alpha|\tau + 1) = \mathrm{e}^{\mathrm{i}\pi((\lambda^2/2N)-1/12)} \mathcal{F}_{\lambda,\lambda+1}(\alpha|\tau) = T_{\lambda\lambda} \mathcal{F}_{\lambda,\lambda+1}(\alpha|\tau) \tag{4.8}$$

and under the transformation S, using the Poisson formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{p=-\infty}^{\infty} \int \mathrm{d}x\, f(x)\mathrm{e}^{2\mathrm{i}\pi px} \tag{4.9}$$

we obtain

$$\mathcal{F}_{\lambda,\lambda+1}\left(\alpha\middle| -\frac{1}{\tau}\right) = \tau^{1/2N}\frac{1}{\sqrt{2N}}\sum_{\lambda'=1-N}^{N} \mathrm{e}^{-\mathrm{i}\pi\frac{\lambda\lambda'}{N}}\mathcal{F}_{\lambda',\lambda'+1}(\alpha\tau|\tau)$$
$$\doteq \tau^{1/2N}\sum_{\lambda'=1-N}^{N} S_{\lambda\lambda'}\mathcal{F}_{\lambda',\lambda'+1}(\alpha\tau|\tau). \qquad (4.10)$$

In other words the action of T is diagonal, being the multiplication by $\exp(2\mathrm{i}\pi\lambda^2/4N - 1/24)$, while the action of S is nothing but the finite Fourier transform over integers modulo $2N$.

To obtain the partition function [11] from the 2-point function $\mathcal{F}_{\lambda,\lambda+1}(\alpha|\tau)$, we proceed as in the previous section, taking the limit $\alpha \to 0$ in (3.11) (expansion around $\alpha = 0$) and compare with the expected value of the operator product expansion of the quantum fields (short-distance limit of $A(\alpha)A(0)$). Since the irreducible representations of Z_{2N} can be labelled by $\lambda = 1-N, 2-N, \ldots, N$ we will denote the corresponding representation space by H_λ. Thus, the chiral partition function of the Z_{2N} conformal models is given by

$$Z_\lambda(\beta) = \mathrm{Tr}_{H_\lambda}(\mathrm{e}^{-\beta L_0}) = \frac{1}{\eta(\tau)}\sum_{l\in Z} q^{N(l+\lambda/2N)^2} \doteq \chi_\lambda(\tau). \qquad (4.11)$$

Therefore, equations (4.8) and (4.10) state that the modular properties of $\mathcal{F}_{\lambda,\lambda+1}(\alpha|\tau)$ coincide with the modular properties of the corresponding characters $\chi_\lambda(\tau)$ (multiplied by $\mathrm{e}^{2\mathrm{i}\pi\alpha}$) [4].

We note that the characters obey the periodic condition

$$\chi_{\lambda+2N}(\tau) = \chi_\lambda(\tau) \qquad (4.12)$$

and

$$\sum_{\lambda'=1-N}^{N} C_{\lambda\lambda'}\chi_{\lambda'}(\tau) = \chi_\lambda(\tau) = \chi_{-\lambda}(\tau) \qquad (4.13)$$

where $C = S^2$ is the Z_{2N}-charge conjugation matrix.

The fact that it is possible to find the characters $\chi_\lambda(\tau)$ from the functions $\mathcal{F}_{\lambda,\lambda+1}(\alpha|\tau)$ suggests that there is a hidden relationship between the KMS matrices $(\tilde{S}, \tilde{T})$ and the matrices (S, T), the representing matrices of the modular group. Actually, using the KMS condition we can write from (4.10)

$$\mathcal{F}\left(\alpha - \frac{1}{\tau}\middle| -\frac{1}{\tau}\right) = \tau^{1/2N} S\mathcal{F}(\alpha\tau - 1|\tau)$$
$$= \tau^{1/2N} S\tilde{T}\mathcal{F}(\alpha\tau|\tau)$$
$$= \tau^{1/2N}(S\tilde{T}^* S^*)S\mathcal{F}(\alpha\tau|\tau). \qquad (4.14)$$

But, since $\mathcal{F}(\alpha - 1/\tau| - 1/\tau) = \tilde{S}\mathcal{F}(\alpha| - 1/\tau)$, we obtain

$$\tilde{S} = S\tilde{T}^* S^*.$$

Similarly

$$\tilde{T} = T\tilde{T}T^*$$

as expected; since $\tilde{T}$ and $\tilde{S}$ act on α and T and S act on τ, the two variables of $\mathcal{F}$, these actions have to commute. Finally we recall (3.13) to remember that the $\tilde{T}$ and $\tilde{S}$ matrices satisfy the relation $(\tilde{S}\tilde{T})^{2N} = \omega^N \mathbf{1}$, where $\omega = \exp(-2\mathrm{i}\pi/2N)$, which is consistent with the modular constraint $(ST)^3 = S^2$.

With these expressions we have achieved our goal to determine the modular properties of the characters in terms of the KMS boundary condition and the centre of the conformal group. From the fusion rules we have also constructed the KMS matrix $\tilde{S}$ (exchange matrix elements) which can be diagonalized by the modular matrix S. We believe that this case gives an illustration of the general case.

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